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APPLICATION OF CONTINUED MATRIX FRACTIONS TO THE ANALYSIS OF STOCHASTIC SYSTEMS WITH POLYNOMIAL NON-LINEARITIES*

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A stochastic system in the form of an oscillator with non-linear stiffness and non-linear damping acted upon by a random Gaussian delta-correlated force is considered. The non-linearities are approximated by polynomials. Vectors whose components are stationary values of moments of the output coordinate are introduced. Using Markov theory, a chain of three-term relations is obtained for them and the solution is represented in the form of a continued matrix fraction. The convergence of these fractions to the exact results is investigated numerically for specific examples.

It is well-known that analytic solutions of the Fokker-Planck equation describing the probable distributions of the output coordinates of non-linear stochastic systems can only be found for a few special cases. Existing approximate methods of statistical description, like the method of statistical linearization, Gaussian approximation and more general cumulant approaches /1-3/, do not always adequately reflect the behaviour of systems under fairly strong random influences. The use of continued matrix fractions was suggested in /4/ for finding the stationary values of the moments of the output coordinates of stochastic systems with cubic non-linearity. (For one-dimensional systems the moments are found in the form of ordinary continued fractions /5/). The possibility of using matrix continued fractions to analyse stochastic systems is demonstrated below. Because the problem of the convergence of such a procedure both in general and for specific dynamic systems is an open question, this method is used here for systems which in certain cases have an exact statistical description.

1. Consider a non-linear oscillator under the influence of a stochastic force

$$x'' + 2h\,(1 + f\,(x))x' + \Omega^2\,(1 + g\,(x))x = \Omega^2\xi\,(t) \tag{1.1}$$

We will assume that the noise is Gaussian and delta-correlated:

$$\langle\xi\rangle = 0,\quad \langle\xi\,(t)\,\xi\,(t-\tau)\rangle = D_\xi\delta\,(\tau)$$

The effects of non-linear rigidity and non-linear damping will be considered separately, although the approach to be used here can, in principle, take both these factors into account. Introducing dimensionless variables, we write Eq.(1.1) in the form

$$\begin{gathered}\Omega^{-1}x' = y,\quad \Omega^{-1}y' = -\delta F\,(x)\,y - G\,(x) + \xi\,(t)\\ F\,(x) = 1 + f\,(x),\quad G\,(x) = x + xg\,(x),\quad \delta = 2h/\Omega = Q^{-}\end{gathered} \tag{1.2}$$

(where Q is the selectivity of the corresponding linear system). We obtain in the standard way the stationary Fokker-Planck equations for the joint probability densities of system (1.2):

$$y\,\frac{\partial W}{\partial x} - \delta F\,(x)\,\frac{\partial}{\partial y}\,(yW) - G\,(x)\,\frac{\partial W}{\partial y} = \delta D\,\frac{\partial^2 W}{\partial y^2},\quad D = \frac{\Omega D_\xi}{2\delta} \tag{1.3}$$

where the parameter D is the mean energy of the Brownian motion of the linear system.

When there are no non-linear losses $(F\,(x) = 1)$ the solution of Eq.(1.3) is known:

$$W\,(x, y) = W_x\,(x)\,W_y\,(y) = C\exp\left[-\frac{1}{D}\left(\frac{y^2}{2} + \int G\,(x)\,dx\right)\right] \tag{1.4}$$

We remark that the probability density of the coordinate $W_x\,(x)$ satisfies a simple ordinary differential equation:

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$$dW_x/dx = -D^{-1}G(x)\,W_x$$

which can be shown to be a stationary Fokker-Planck equation for the first-order stochastic system

$$\Omega_1^{-1}x' + G(x) = \xi(t), \quad \Omega_1 = Q\Omega \tag{1.5}$$

Hence, when there are no non-linear losses, Eq.(1.5) is statistically equivalent to the original system (1.1) for finding the stationary characteristics of the coordinate.

2. We will assume tht the functions $f(x)$ and $g(x)$ are specified by convergent power series; from physical considerations $f(x)$ should be even, as should the last remaining term in the expansion of $g(x)$.

We will first assume that there are no non-linear losses and that the non-linear stiffness is an even function given by the expansion

$$g(x) = \sum_{k=1}^{m} b_k x^{2k} \tag{2.1}$$

(which corresponds to Brownian motion with a symmetric potential). Putting $x^{2k} = I^k$, we obtain from relations (1.5) and (2.1) an infinite chain of relations for the stationary moments of the intensity:

$$\langle I^s\rangle + \sum_{k=1}^{m} b_k \langle I^{s+k}\rangle = (2s-1)\,D\,\langle I^{s-1}\rangle, \quad s = 1, 2, \ldots \tag{2.2}$$

Introducing moment vectors of dimension m of the form

$$\mathbf{X}_1 = (\langle I\rangle, \langle I^2\rangle, \ldots, \langle I^m\rangle), \ldots$$
$$\mathbf{X}_n = (\langle I^{(n-1)m+1}\rangle, \langle I^{(n-1)m+2}\rangle, \ldots, \langle I^{nm}\rangle)$$

we can change from Eqs.(2.2) to the corresponding matrix chain of three-term relations

$$A_1\mathbf{X}_1 + B_1\mathbf{X}_2 = C_1\mathbf{X}_0, \ldots, A_n\mathbf{X}_n + B_n\mathbf{X}_{n+1} = C_n\mathbf{X}_{n-1} \tag{2.3}$$

where the auxiliary vector $\mathbf{X}_0$ has components $(0, \ldots, 0, 1)$. The form of the matrices in the chain (2.3) is found from Eqs.(2.2). In particular, restricting expansion (2.1) to two terms, we have

$$\mathbf{X}_n = (\langle I^{2n-1}\rangle, \langle I^{2n}\rangle), \quad A_n = \begin{Vmatrix} 1 & b_1 \\ -(2n+1)D & 1 \end{Vmatrix} \tag{2.4}$$
$$B_n = \begin{Vmatrix} b_2 & 0 \\ b_1 & b_2 \end{Vmatrix}, \quad C_n = \begin{Vmatrix} 0 & (2n+1)D \\ 0 & 0 \end{Vmatrix}$$

while for $m = 3$

$$A_n = \begin{Vmatrix} 1 & b_1 & b_2 \\ D(3-6n) & 1 & b_1 \\ 0 & D(1-6n) & 1 \end{Vmatrix}, \quad B_n = \begin{Vmatrix} b_3 & 0 & 0 \\ b_2 & b_3 & 0 \\ b_1 & b_2 & b_3 \end{Vmatrix} \tag{2.5}$$

(the only non-zero element in the matrix C_n being $C_{13} = D(6n-5)$).

The solution of the chain (2.3) for the required moment vector has the form of a continued matrix fraction

$$\mathbf{X}_1 = \cfrac{C_1\mathbf{X}_0}{A_1 + \cfrac{B_1C_2}{A_2 + \cfrac{B_2C_3}{A_3 + \ldots}}} \tag{2.6}$$

and the appropriate computational algorithm can be easily implemented on a computer, (for details see /4/).

3. As an initial example we will consider Brownian motion of systems in potential wells of the form

$$G_1(x) = \mathrm{sh}(x), \quad G_2(x) = \mathrm{tg}(x) \tag{3.1}$$

To use the above method we will confine ourselves to the first two non-linear terms of the expansions,

$$G_1(x) \simeq x + \frac{x^3}{3!} + \frac{x^5}{5!}, \quad G_2(x) \simeq x + \frac{x^3}{3} + \frac{2x^5}{15} \tag{3.2}$$

We shall compare the first two moments of the intensity obtained from the continued fraction (2.6) with the results of numerical integration of the corresponding probability distributions.

Table 1

D	Non-linearity G_1			Non-linearity G_2		
	1	2	4	0,25	0,5	1
$\langle I \rangle$	0,731 0,732 0,732	1,223 1,230 1,229	1,936 1,960 1,961	0.197 0.202 0,202	0.322 0.341 0.346	0.467 0.505 0.559
$\langle I^2 \rangle$	1,396 1,404 1,403	3,723 3,778 3.780	8.894 9,189 9,181	0,102 0,109 0,109	0.250 0.290 0.304	0,479 0,569 0,749
n	4	5	7	4	6	9

Table 2

	$D = 0.5$				$D = 2$			
a_2	$a_1 = 0$	0,125	0,5	2	0	0,125	0,5	2
0	1	2	4	6	1	4	8	14
0,125	4	2	4	5	20	15	9	8
0,5	6	9	5	5	*	*	19	6
1	10	12	9	5	*	*	*	11

Comparative data are given in Table 1. Here the first rows give moments found by numerical integration of the probability distribution (1.4) with non-linearities of the form (3.1), the second rows give the corresponding results for numerical integration with polynomial non-linearities (3.2), and the third rows give results from the continued fraction (2.6) using approximation (3.2). The final row gives the number of approximations (equal to the order of the fraction approximating (2.6) for which the required accuracy of 10^{-3} was achieved). We note that for non-linearity G_1 the second and third rows are practically identical. The relatively large difference between the second and third rows for non-linearity G_2 is due to the fact that the corresponding probability distribution

$$W_x(x) = C(\cos x)^{1/D}, \quad -\pi/2 < x < \pi/2$$

has compact support and the moments in the second rows were also obtained by integrating over this interval. By integrating over an "infinite" interval the difference between these rows did not exceed 0.003 for the given values of D. The slower convergence of the continued fraction for the non-linearity G_2 is associated with the larger expansion coefficients in (3.2).

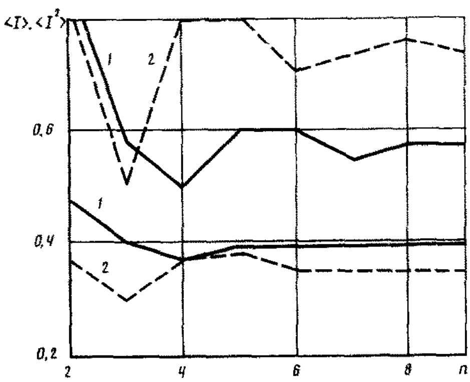

Fig.1

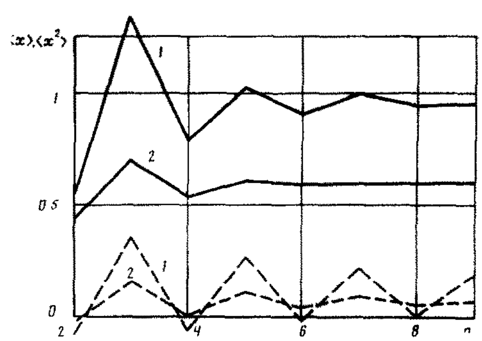

Fig.2

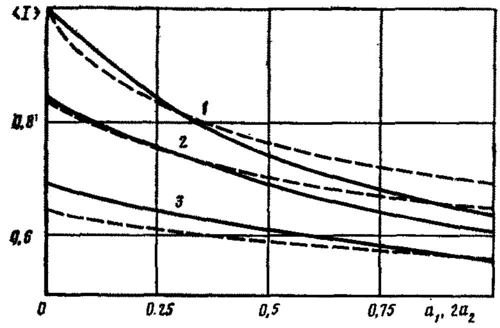

Fig.3

On the basis of a computer analysis one can conclude that for this problem the continued fractions converge for any sensible values of the cubic non-linearity coefficient b_1 and effective strength of the input noise D, although there is a restriction on the coefficient b_2 which should be less than unity and should not exceed b_1. Fig.1 shows the convergence history of the method for a very large non-linearity $(b_1 = 1, b_2 = 0.5)$. Here n is the number of iterations, the solid curve shows the value of the mean intensity, and the dashed curve shows the mean square intensity. The lower curves are for $D = 1$ and the upper ones are for $D = 2$.

4. We will now consider a non-symmetric potential well, assuming that the expansion of the function $g(x)$ contains all powers from 1 to $2m$. The non-symmetry leads to a non-zero odd moments of the coordinate, so we introduce the moment vectors as follows:

$$\begin{aligned} \mathbf{X}_1 &= (\langle x\rangle, \ldots, \langle x^{2m-1}\rangle, \langle x^{2m}\rangle), \ldots \\ \mathbf{X}_n &= (\langle x^{2(n-1)m+1}\rangle, \ \langle x^{2(n-1)m+2}\rangle, \ldots, \langle x^{2n-m}\rangle) \end{aligned} \tag{4.1}$$

The chain of relations for stationary moments analogous to (2.2) is

$$\langle x^s\rangle + \sum_{k=1}^{2m} b_k \langle x^{s+k}\rangle = (s-1) D \langle x^{s-2}\rangle, \quad s = 1, 2, \ldots \tag{4.2}$$

$\mathbf{X}_1$ can be represented in the form (2.6), the matrices being found using (4.1) and (4.2). In particular, using only two terms in the expansion of the non-linearity $(m = 1)$ we obtain

$$\mathbf{X}_n = (\langle x^{2n-1}\rangle, \langle x^{2n}\rangle), \quad n = 1, 2, \ldots \tag{4.3}$$

$$A_n = \begin{Vmatrix} 1 & b_1 \\ 0 & 1 \end{Vmatrix}, \quad B_n = \begin{Vmatrix} b_2 & 0 \\ b_1 & b_2 \end{Vmatrix}, \quad C_n = D \begin{Vmatrix} 2n-2 & 0 \\ 0 & 2n-1 \end{Vmatrix}$$

It is clear that the even moments are of interest, while the odd moments play the role of correction terms associated with the a symmetry of the potential. Numerical analysis shows that the method converges for all reasonable values of the cubic non-linearity coefficient b_2, whereas the coefficient b_1 should be less than unity and should not exceed b_2. (The sign of b_1 influences the sign of the odd moments and does not influence the even ones).

Fig.2 shows the convergence history for the continued fraction similar to Fig.1. The solid curves are mean square coordinates and the dashed curves are mean values. We note that the mean values converge to the exact result much more slowly than $\langle x^2\rangle$.

5. Consider the influence of non-linear damping, assuming that the stiffness is now linear $(g(x) = 0)$. We approximate the function $f(x)$ by the expansion

$$f(x) \simeq a_1 x^2 + a_2 x^4 \tag{5.1}$$

Assuming the system to be selective, we change to the amplitude and phase of the oscillations in the standard way and during statistical averaging we perform an averaging over fast time (see, e.g., /2, 6/). The statistical characteristics of the amplitude decouple from those of the phase and one can obtain the following chain of equations for the moments of the intensity, $(I = {}^1/_2 A^2$, where A is the amplitude):

$$\langle I^s\rangle + {}^1/_2 a_1 \langle I^{s+1}\rangle + {}^1/_2 a_2 \langle I^{s+2}\rangle = sD \langle I^{s-1}\rangle \tag{5.2}$$

The matrix description of this system is little different from formulae (2.4): the B_n matrices are identical with those in (2.4), where $b_k = {}^1/_2 a_k$, $k = 1, 2$, while the others have the form

$$A_n = \begin{Vmatrix} 1 & {}^1/_2 a_1 \\ -2nD & 1 \end{Vmatrix}, \quad C_n = \begin{Vmatrix} 0 & (2n-1) D \\ 0 & 0 \end{Vmatrix}$$

Table 2 shows how the convergence of the results depends on a_1, a_2 and D. It gives the number of iterations required to give an absolute accuracy of 10^{-3} for the mean intensity. (As was shown in /5/, for a purely cubic non-linearity $(a_2 = 0)$ the corresponding one-dimensional continued fraction always converges). The symbol * indicates that the required accuracy was not achieved. As can be seen from the table, the convergence slows down as the effective noise strength D and coefficient a_2 increase. The convergence also worsens for $a_2 \geqslant a_1$, but such a situation is physically unnatural.

Fig.3 shows the dependence of the mean intensity of the oscillations on the coefficient a_1 (the solid curves) and a_2 (the dashed curves) for $D = 1$. For the given parameter values the continued fractions converged to the exact results. The exact values of the moments can be found by integrating the probability distribution of the intensity

$$W(I) = C \exp[-D^{-1} I (1 + a_1 I + {}^4/_3 a_2 I^2)]$$

6. In conclusion we point out the possibility of applying this method directly to the stochastic Eq.(1.1) if the non-linear functions $f(x)$ and $g(x)$ are given by expansions of the form (2.1). A closed statistical description is possible only for a Markov set $\{x, y\}$. The chain of equations for the stationary values of the joint moments $\langle p; q\rangle \equiv \langle x^p y^q\rangle$ has the form

$$\begin{gathered} qQ\langle p+1; q-1\rangle + pQ\langle p-1; q+1\rangle + q\langle p; q\rangle + \\ q\sum_{k=1}^{m} (a_k\langle p+2k; q\rangle + Qb_k\langle p+2k+1; q-1\rangle) = q(q-1) D\langle p+1; q-2\rangle \end{gathered} \tag{6.1}$$

where the indices p and q are positive integers; their sum must be even because of the vanishing of all odd moments. (Moments of the form $\langle p; 1\rangle$ are also, of course, zero). The original moment vector $\mathbf{X}_1$ here has components

$$\begin{gathered} \mathbf{X}_1 = (\langle x^2\rangle, \langle y^2\rangle; \langle x^4\rangle, \langle x^2y^2\rangle, \langle xy^3\rangle, \langle y^4\rangle; \ldots \\ \langle x^{2m}\rangle, \langle x^{2m-2}y^2\rangle, \langle x^{2m-3}y^3\rangle, \ldots, \langle y^{2m}\rangle) \end{gathered} \tag{6.2}$$

and dimensionality equal to

$$l_1 = 2 + 4 + \ldots + 2m = m(m+1)$$

It follows from (6.1) that the dimensionality of vectors of higher orders, introduced in the same way as $\mathbf{X}_1$, is as follows:

$$l_2 = m(3m+1), \ldots, l_n = m[(2n-1)m+1]$$

The solution for the vector X_1 has the form (2.6). The matrices A_n, B_n and C_n are determined from (6.1), and their dimensionality now increases as the order increases. A similar problem was previously solved /4/ for the case of a purely cubic non-linearity $(m = 1)$.

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